Multilevel Image Reconstruction Using Natural Pixels
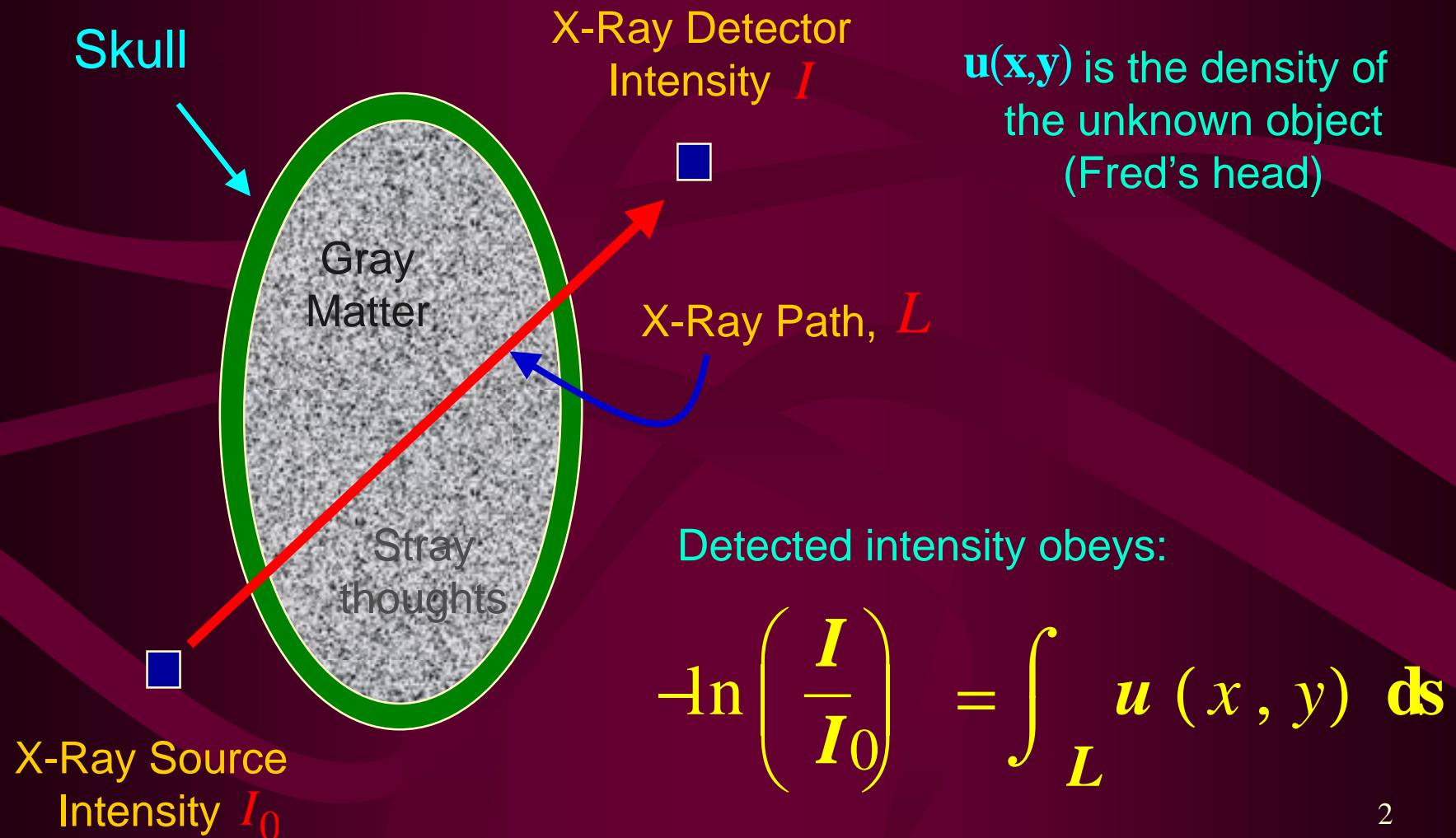
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In association with:
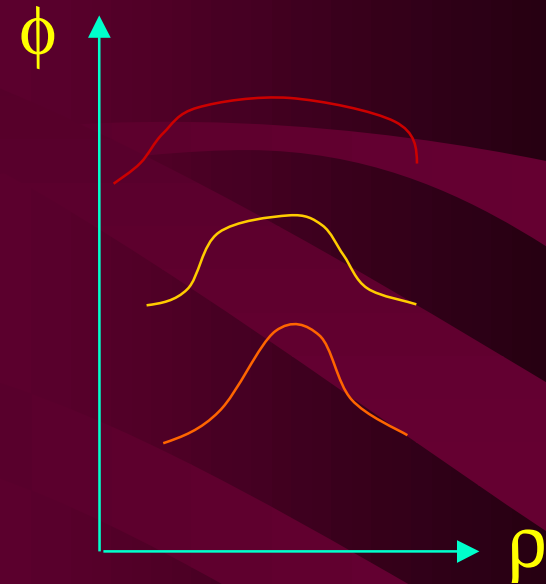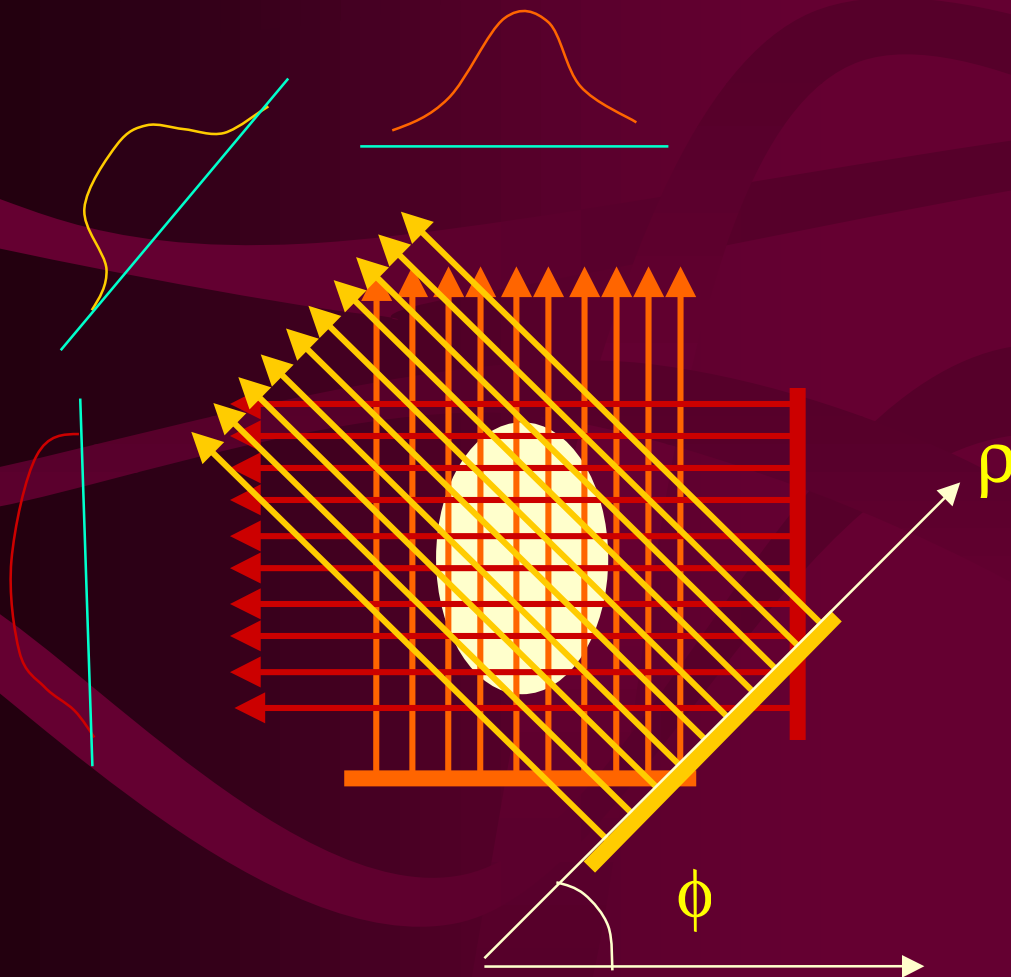
Mark A. Limber   Bruce T. Robinson   Stephen F.McCormick

Auto-Trol Inc   Accurate Information Systems   University of Colorado

Fred's Head

Skull

X-Ray Detector Intensity I

$\mathbf{u(x,y)}$ is the density of the unknown object (Fred's head)

Gray Matter

X-Ray Path, L

Stray thoughts

Detected intensity obeys:

$$-\ln\left(\frac{I}{I_0}\right) = \int_L u(x,y)\ \mathbf{ds}$$

X-Ray Source Intensity I_0

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The Radon Transform

$$R[u] = \int_{\mathbf{R}^2} u(x,y)\,\delta(\rho - x\cos\phi - y\sin\phi)\,dx\,dy$$

Can we recover $u(x,y)$ from a sampling of $R[u]\,(\rho,\phi)$?

Reconstruction Techniques

Image reconstruction techniques fall into two categories:

- Direct methods based on the Central Slice Theorem:

  - The 1-d Fourier Transform of each view in the Radon Transform is a "slice" through the 2-d Fourier Transform of the unknown function.

- Iterative methods based on discretizing the reconstruction problem as a matrix equation $Rx = b$ and applying an iterative solution method.

  - Different methods result from different methods of discretization and from different choices of iterative methods.

Algebraic Reconstruction Technique

assume $u(x,y) = \sum \alpha_j \phi_j(x,y)$

$$\phi(x,y) = \begin{cases} 1 \text{ if } (x,y) \text{ is in } j \text{ th pixel} \\ 0 \text{ if } (x,y) \text{ not in } j \text{ th pixel} \end{cases}$$

a_{jk} is contribution of jth pixel in computing k^{th} line integral

1	2	3							

n^2
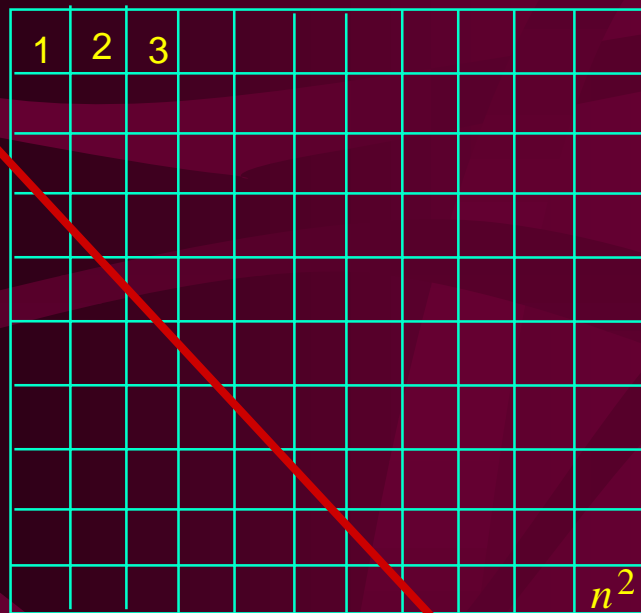
*K*th x-ray

$$[Ru]_k \approx \sum a_{kj} \alpha_j \equiv b_k$$
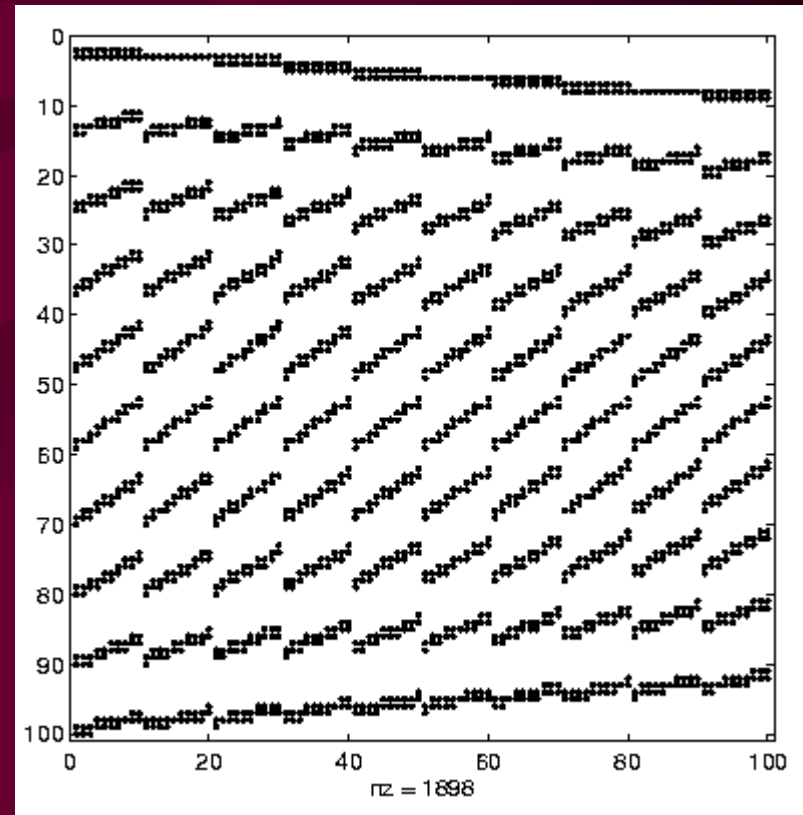
Algebraic Reconstruction Technique
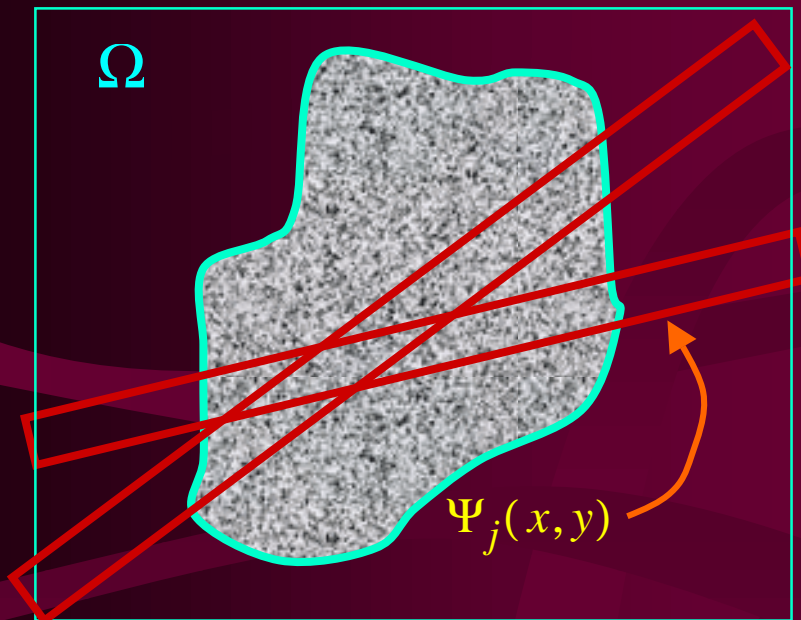
Leads to a large, sparse system

. $$K\,u \; = \; f$$

Sparsity pattern shown for K, where the discretized image is 10 x 10, $M = 20$ views are used, and $n = 5$ strips cover each view.

Matrix is 100 x 100. One row for each of the Mn strips, one column for each of the 10 x 10 pixels.

Natural Pixel Discretization

Ω

$\Psi_j(x,y)$

Let $\Psi_1(x,y)$, $\Psi_2(x,y)$, … $\Psi_N(x,y)$ be characteristic functions of the strips covered by the x-rays (e.g., $\Psi_j(x,y)$ shown at left).
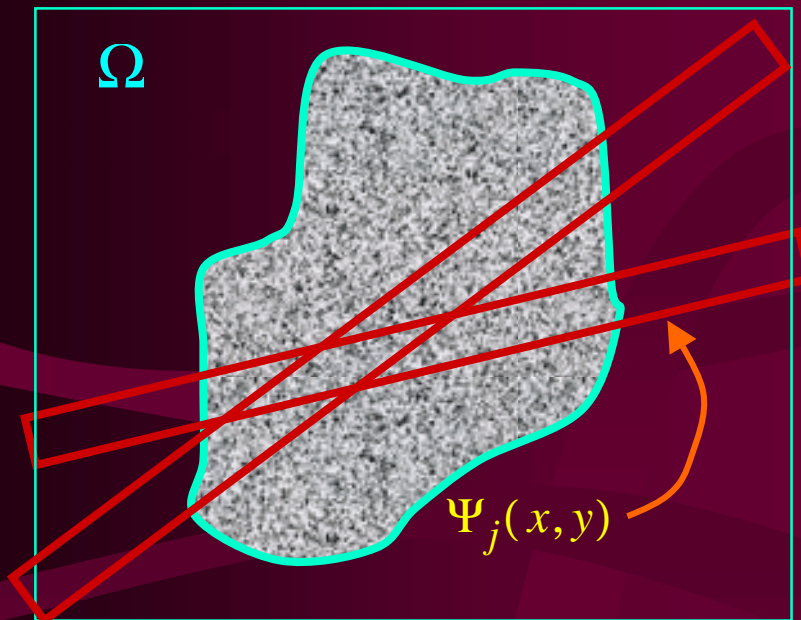
Let $u(x,y)$ be supported in some region Ω , <u>covered</u> by the strips.

Define: $A{:}L_2(\Omega) \Rightarrow \boldsymbol{R}^N$ **by** $(Au)_j = \displaystyle\int_\Omega u(x,y)\,\Psi_j(x,y)\,dx\,dy \equiv \langle \Psi_j, u \rangle$

$$Au = \begin{bmatrix} \langle \Psi_1, u \rangle \\ \langle \Psi_2, u \rangle \\ \dots \\ \langle \Psi_N, u \rangle \end{bmatrix}.$$

We refer to *Au* as the "strip-averaged" Radon Transform.

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Natural Pixel Discretization

Ω

$\Psi_j(x, y)$

The adjoint is given by:

$$\langle Au, f \rangle = \sum f_j \langle \Psi_j, u \rangle$$

$$= \langle \sum f_j \Psi_j, u \rangle \equiv \langle u, A^* f \rangle$$

$$A^* f = \sum f_j \Psi_j(x, y)$$

The action of the adjoint is to "spread" the values of the f_j back along the integral strips from which they came.

Hence $A^* f$ is a backprojection operator!

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Solving $Au = f$

$$A : L_2(\Omega) \Rightarrow R^N$$

A is a short, fat, "matrix", and the function $\Psi_j(x,y)$ is the *j*th "row" of *A*:

$$\begin{bmatrix} ----- \Psi_1(x,y) ----- \\ ----- \Psi_2(x,y) ---- \\ \cdots \\ ----- \Psi_N(x,y) ---- \end{bmatrix}_{N \times \infty} \begin{bmatrix} \\ u \\ \end{bmatrix}_{\infty \times 1} = \begin{bmatrix} \\ f \\ \end{bmatrix}_{N \times 1}$$
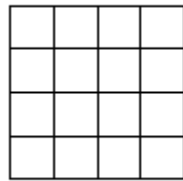
Highly underdetermined, infinitely many solutions, so we seek the minimum norm solution, I.e., seek $w \in R^N$ such that $u = A^* w$ and $Au = f$. Such w gives minimum norm solution.

Observe: Any function $u = A^* w$ will be constant on the polygons defined by the intersections of the strip pixels.
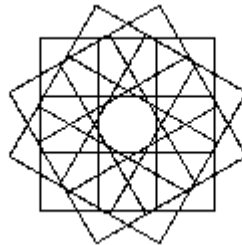
Optimal Grids

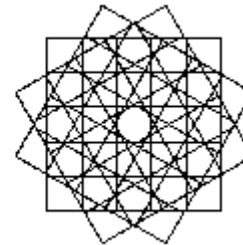Examples of "optimal" grids for simple geometries:

2 views,
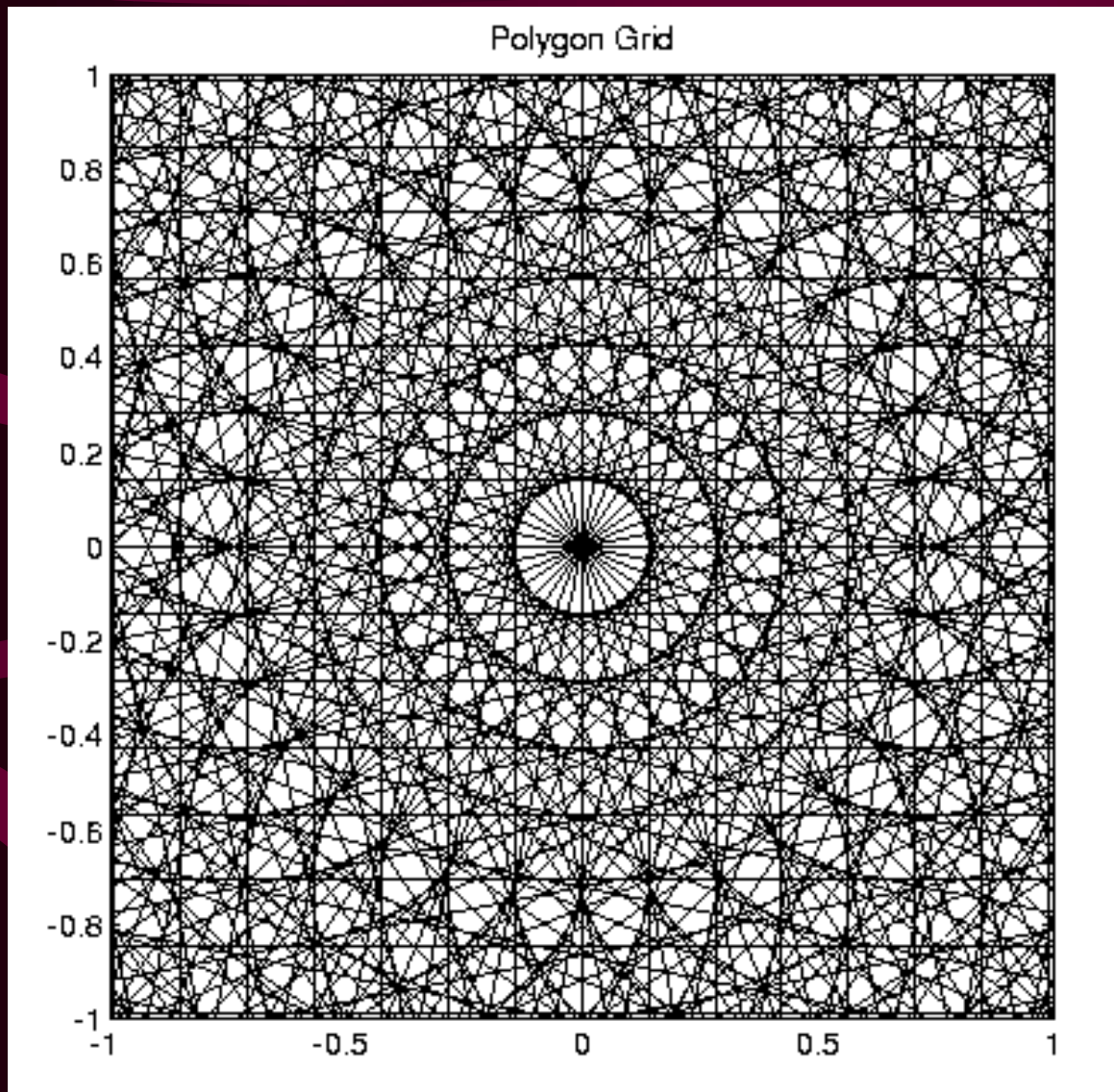4 strips

6 views,
3 strips

6 views,
5 strips

If Ω is the convex hull of the set of strips, the problem is "unconstrained" (and retains some useful symmetries).

If Ω is a square (or other regular region) *inside* the support of the strips, the problem is "constrained."

The Optimal Grid

Polygon Grid

Example optimal grid.

M = 32 (no. of views)

n = 32 (strips per view)

$N = 1024$

Solution satisfies $A(A^*w) = f$

Let $B \equiv AA^*$

$$B \equiv AA^* = \begin{bmatrix} \langle \Psi_1, \Psi_1 \rangle & \langle \Psi_1, \Psi_2 \rangle & \ldots & \langle \Psi_1, \Psi_N \rangle \\ \langle \Psi_2, \Psi_1 \rangle & \langle \Psi_2, \Psi_2 \rangle & \ldots & \langle \Psi_2, \Psi_N \rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle \Psi_N, \Psi_1 \rangle & \langle \Psi_N, \Psi_2 \rangle & \ldots & \langle \Psi_N, \Psi_N \rangle \end{bmatrix}$$

So the equation we wish to solve is:

$$Bw = f$$

Theorems about the matrix B

- B_{ij} is the area of the intersection of the *i*th and *j*th strip pixels.

- *B* is non-negative, symmetric, positive semidefinite.

- If there are *M* views, each with *n* parallel strips, then *N=Mn* and *B* has block structure, where the (*j*,k)th block is *n*x*n* and gives the areas of intersection of the strips of the *j*th and *k*th views :

$$B = \begin{bmatrix} B_{11} & B_{12} & ... & B_{1M} \\ B_{21} & B_{22} & ... & B_{2M} \\ ... & ... & ... & ... \\ B_{M1} & B_{M2} & ... & B_{MM} \end{bmatrix}$$

- B_{ii} is diagonal, and the diagonal entry b_{ii} is the area of the *i*th strip pixel.

- In any block B_{ij} , the sum of the entries on a row equals the entry in that row of the diagonal block
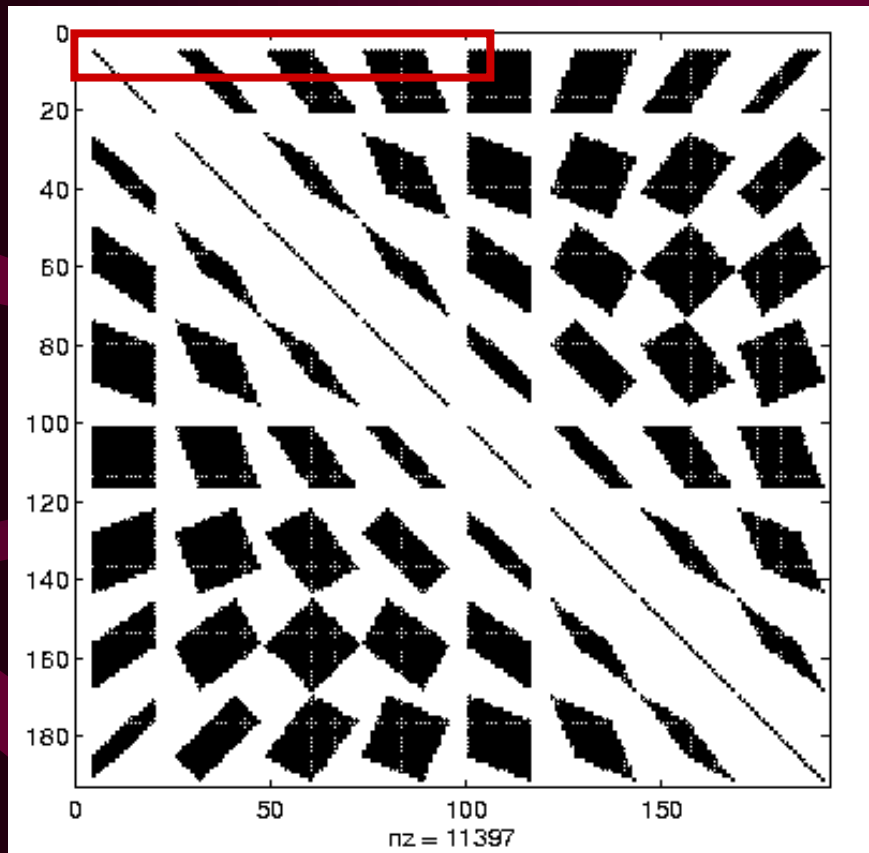
Sparsity and Structure of B
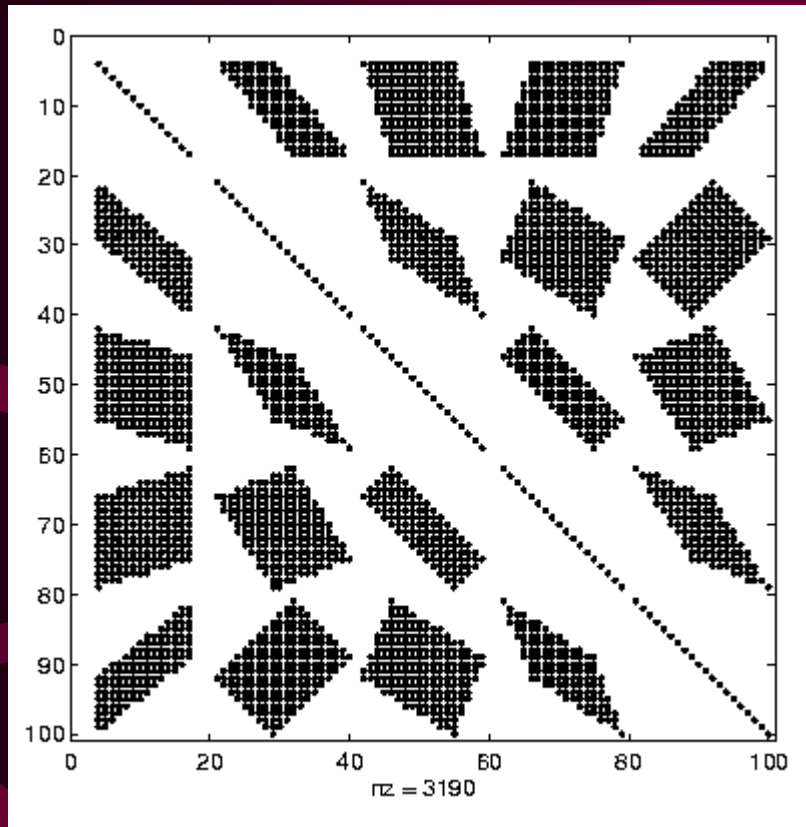
$M = 5$ views, $n = 20$ strips per view. $N = 100$. Matrix is approximately 32% nonzero.

$M = 8$ views, $n = 24$ strips per view. $N = 192$. Matrix is approximately 30% nonzero.

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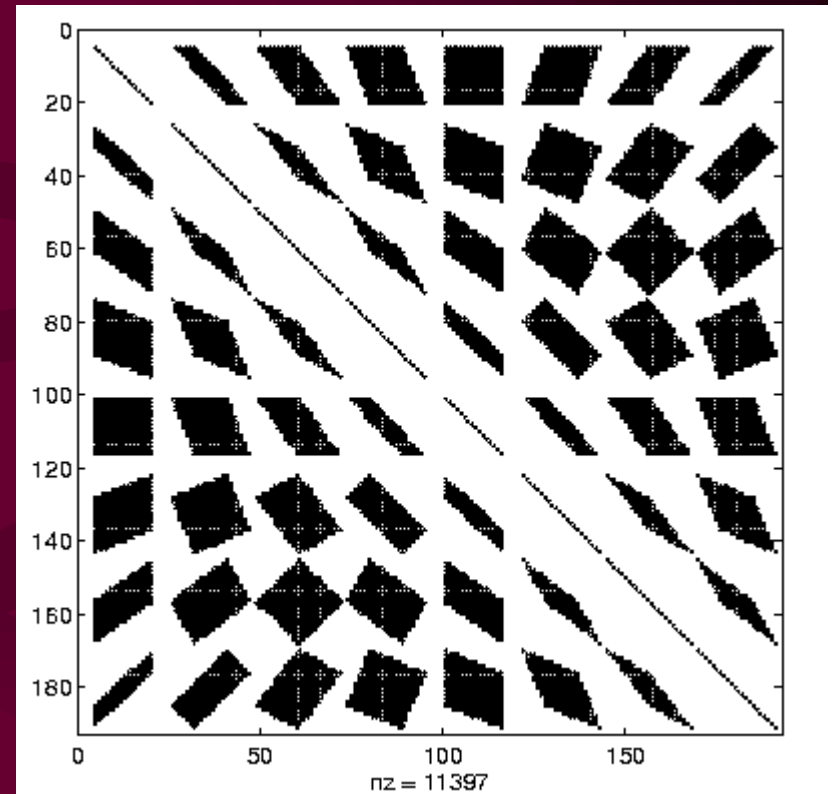
Sparsity and Unconstrained Symmetry

$M = 8$ views, $n = 24$ strips per view. $N = 192$. Matrix is approximately 30% nonzero.

For the unconstrained (convex hull) case with M views and n uniform strips per view, all $(Mn)^2$ entries of the matrix are known from The first $n/2$ rows in the first $(M/2+1)$ blocks.

Constant by Angle

Def: A vector v is constant-by-angle if

$$v = \begin{bmatrix} \alpha_1 \, \alpha_1 \, \alpha_1 \dots \alpha_1 \alpha_2 \, \alpha_2 \, \alpha_2 \dots \alpha_2 \, \alpha_3 \, \alpha_3 \dots \alpha_M \, \alpha_M \dots \alpha_M \end{bmatrix}^T$$

i.e., where all the entries in *v* corresponding to a given view (angle) are constant.

- $v \in NS(B) \ \ iff \ \ v$ is constant-by-angle and $\sum \alpha_j = 0$.

- A basis for $NS(B)$ is given as shown, where the entries +1 and -1

- $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ \dots \\ 0 \end{bmatrix} \dots \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \dots \\ -1 \end{bmatrix}$ represent *n*-vectors of all ones or minus ones,

- corresponding to entire views with those values.

- $dim\,NS(B) \ = M - 1$ and $rank(B) \ = N - (M - 1)$

- Let β_k be the sum of the entries in v corresponding to the *k*th view (angle). If $\beta_k = \beta_j$ for all *j* and *k* then $v \in \ range(B)$
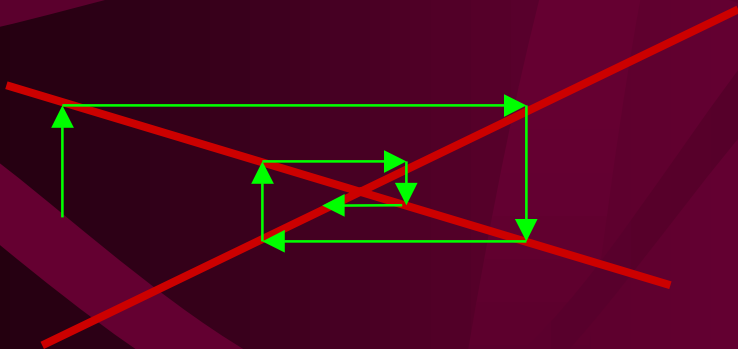
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Gauss-Seidel & Kaczmarz

Gauss-Seidel iteration:

1) Find α such that

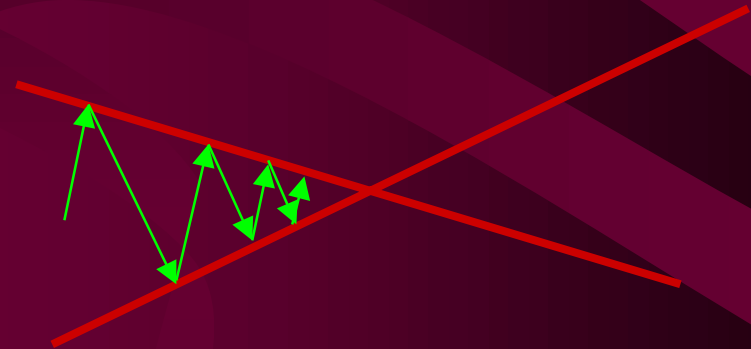$$\left\langle\, e_j,\, A(\,u + \alpha e_j\,) - f\, \right\rangle = 0$$

2) Set $\;u \leftarrow u + \alpha e_j$

Kaczmarz iteration:

1) Find α such that

$$\left\langle\, e_j,\, A(\,u + \alpha A^* e_j\,) - f\, \right\rangle = 0$$

2) Set $\;u \leftarrow (\,u + \alpha A^* e_j\,)$

Solution Methods (1)

- Kaczmarz iteration on $Au = f$, where $u = \sum \alpha_j \Psi_j$

    - Converges moderately well, to minimum norm solution if *f* is in range of *A.* Solution defined on natural pixel grid. Requires knowledge of *B.*

- Kaczmarz iteration on $Au = f$, where $u = \sum \alpha_j P_j(x,y)$ and $P_j(x,y)$ is the characteristic function of the *j*th polygon on the optimal grid.

    - Converges moderately well to minimum norm solution if right-hand side *f* is in range of *A.*

    - *Very* expensive to implement.

- Kaczmarz iteration on $Bw = f,$

    - Very slow to converge, converges to minimum norm solution if right-hand side is in range of *B.*

- Gauss-Seidel on $Bw = f$!
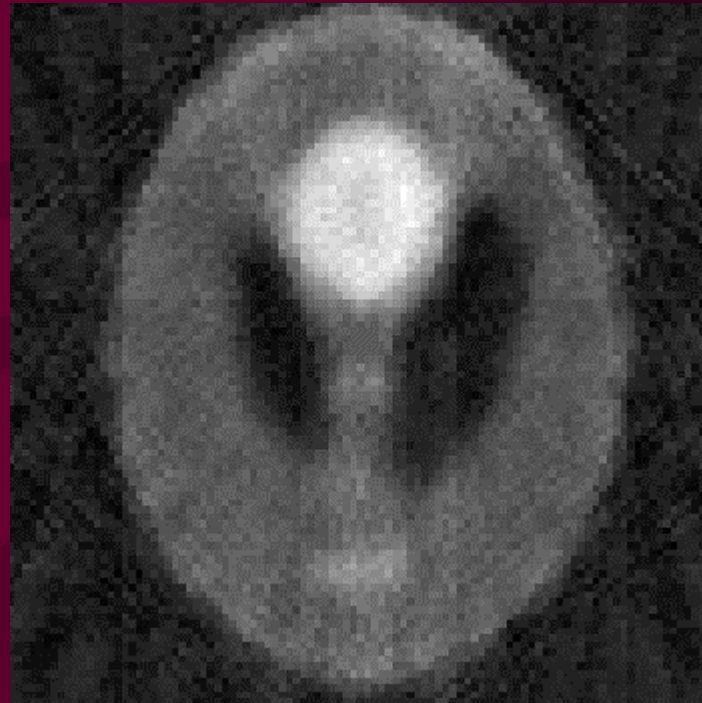
Gauss-Seidel on $Bw = f$.

Theorems about GS on $Bw = f$:

- GS on $Bw = f$ cannot diverge in the energy seminorm, e.g. if w^* solves $Bw = f$ then $|||w^{(n+1)} - w^*||| \leq |||w^{(n)} - w^*|||$ where $|||v||| \equiv \langle Bv, v \rangle$.

- Let $w^{(n+1)} \leftarrow GS(w^{(n)})$ be (n+1)st sweep of GS on $Bw = f$.

  Let $u^{(n+1)} \leftarrow Kacz(u^{(n)})$ be (n+1)st sweep of Kaczmarz on $Au = f$.

  If $u^{(1)} = A^* w^{(1)}$ then $u^{(n+1)} = A^* w^{(n+1)}$.

- If f is in range(B) then GS converges to w such that $u = A^* w$ is the minimum 2-norm solution to $Au = f$.

- $\rho(GS) \leq 1$, and if $GSv = \lambda v$, $\|\lambda\| = 1$, then $v \in NS(B)$.

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A Gauss-Seidel Reconstruction

"Exact" image, used to create projection set, with 64 uniform-width strips at each of 20 angles. Matrix is 1280 x 1280.
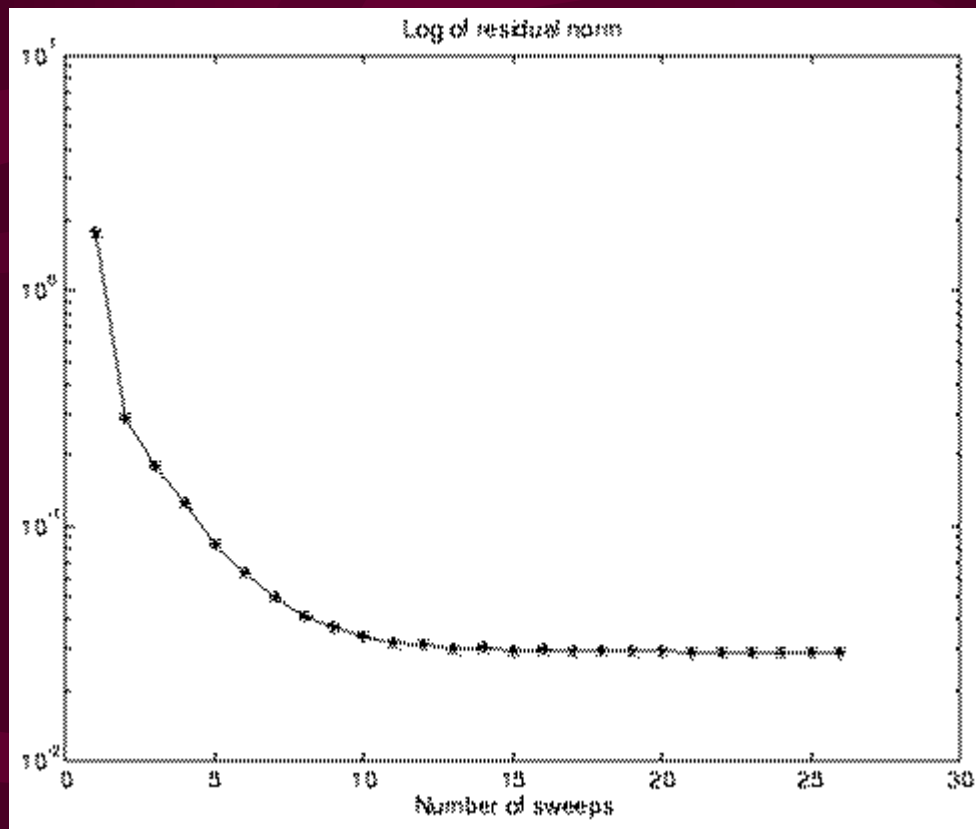
Reconstruction using 25 sweeps of Gauss-Seidel iteration on $Bw = f$.

Gauss-Seidel Performance

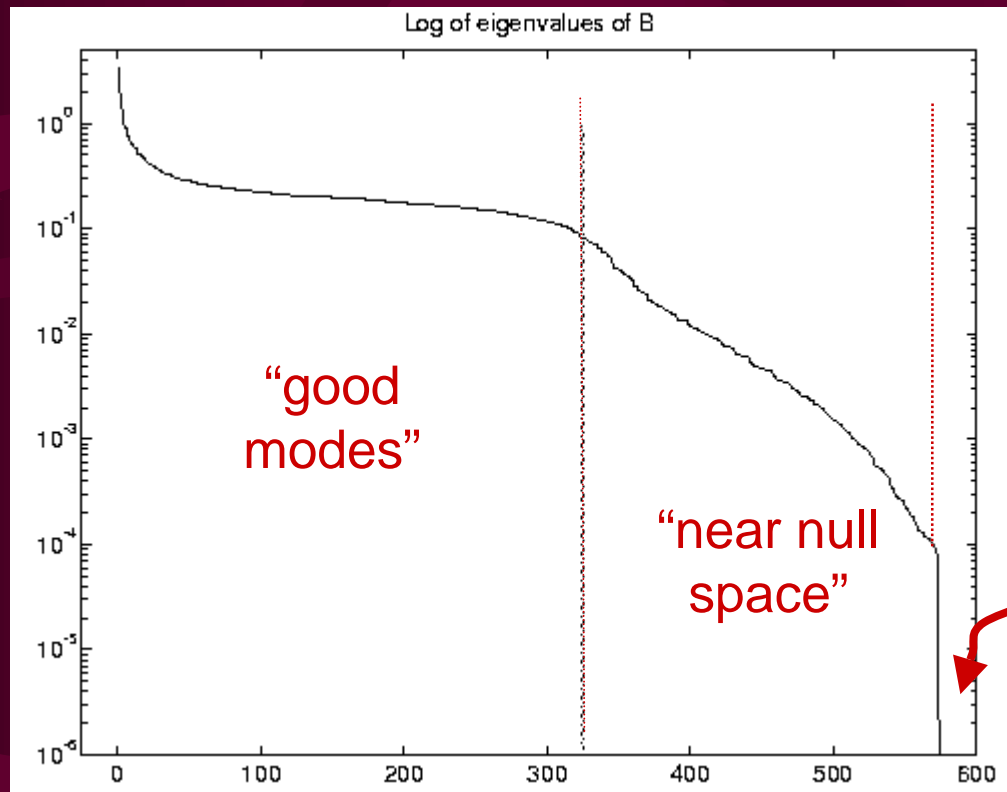Logarithm of norm of the residual, $\left\| f - Bw^{(n)} \right\|_2$ plotted as a function of the number of iteration sweeps.
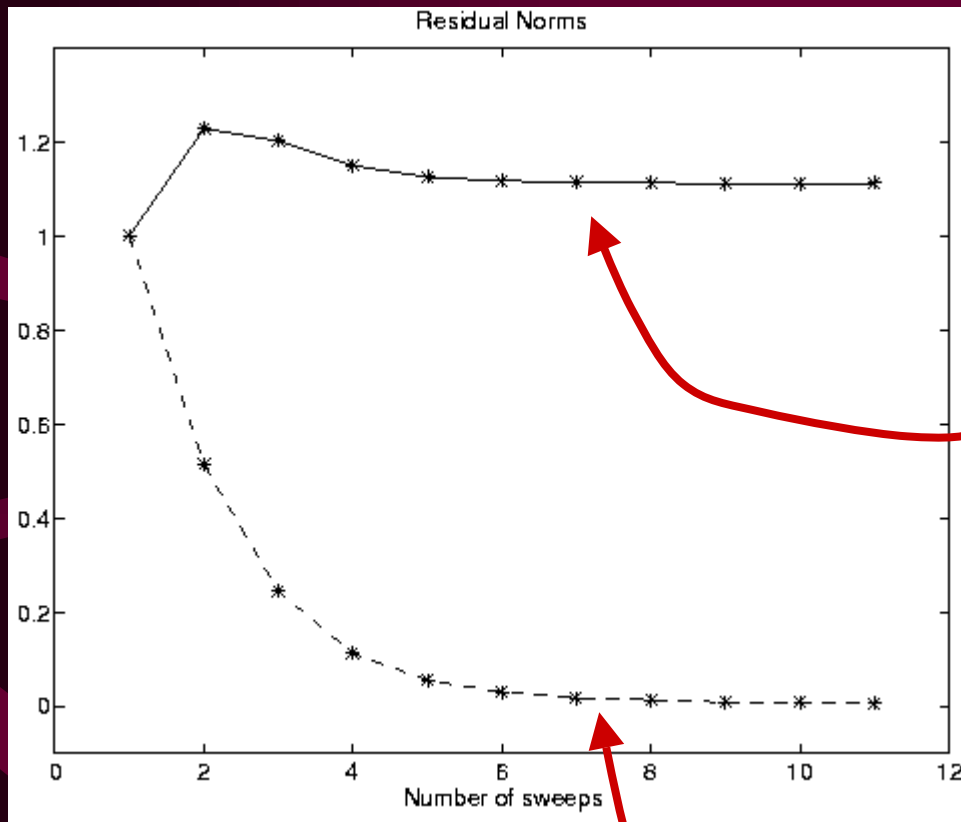
The iteration stalls after a few GS sweeps.

Spectral Analysis of *B*

Logarithm of the eigenvalues. *B* is 592 x 592, with *M* = 20. Components of the error from the "near null space" are slow to converge under Gauss-Seidel iteration.

Log of eigenvalues of B

"good modes"

"near null space"

Null space: dimension is 19

GS on "good" and "bad" modes

Residual norm of $Bw = 0$

as a function of GS sweeps.

Initial guess = v_{540}

Initial guess = v_{15}

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Gauss-Seidel on a "good" mode

Spectral decomposition of result of one GS sweep on $Bw = 0$, using the eigenvector v_{15} as initial guess.

Spectral plot, mode 15

Good modes
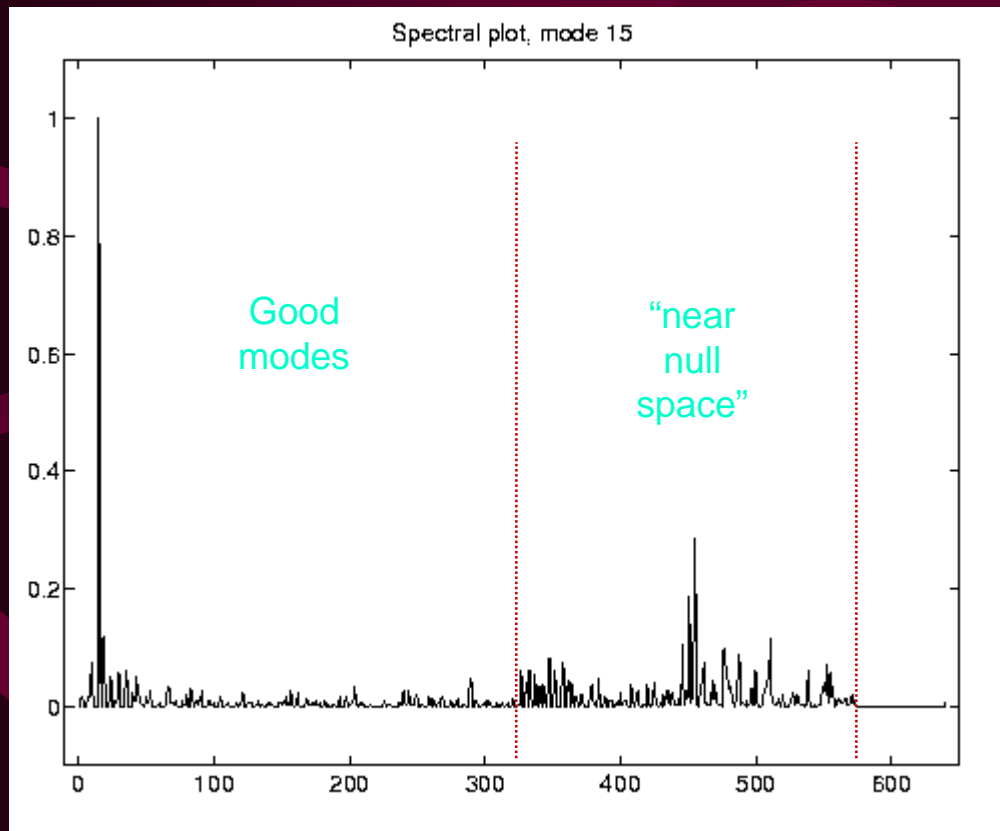
"near null space"

For the "good" mode, GS mixes modes moderately, by exciting minor contributions from modes in the "near null space."

Gauss-Seidel on a "bad" mode

Spectral decomposition of result of one GS sweep on $Bw = 0$, using the eigenvector v_{540} as initial guess.

Spectral plot; mode 540

Good modes
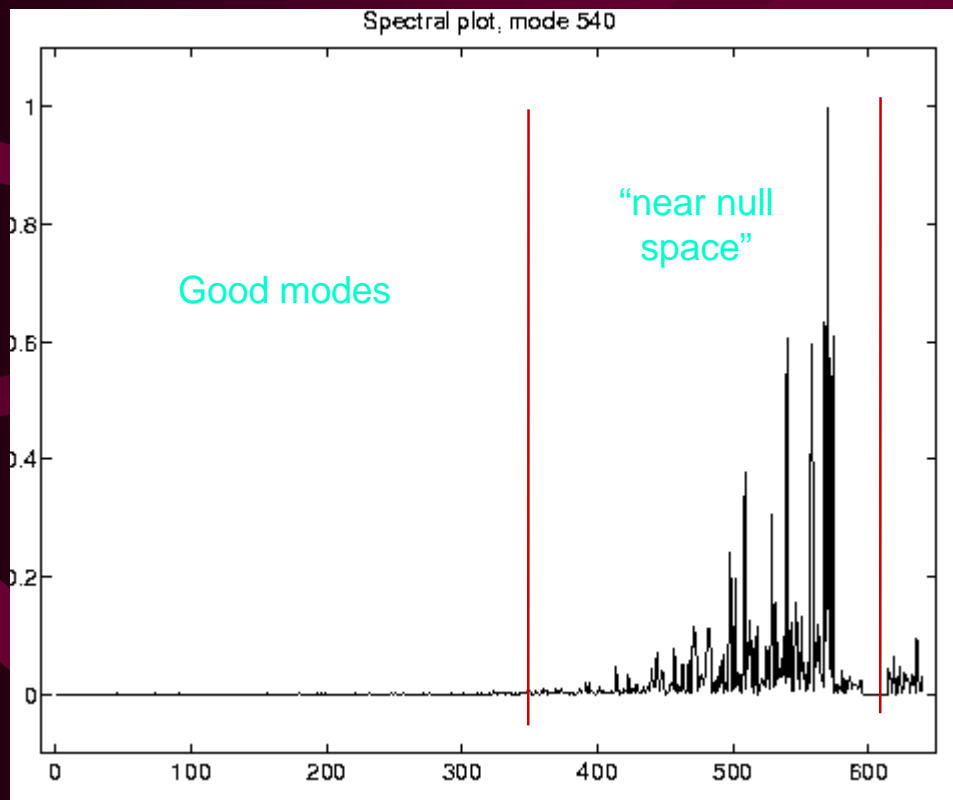
"near null space"

For the "bad" mode, GS mixes modes severely, by exciting major contributions from other modes in the "near null space."
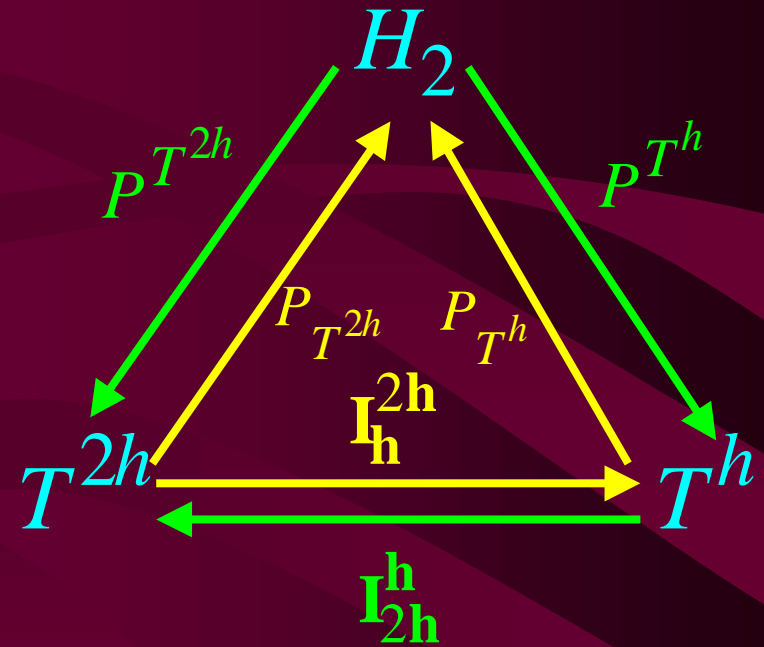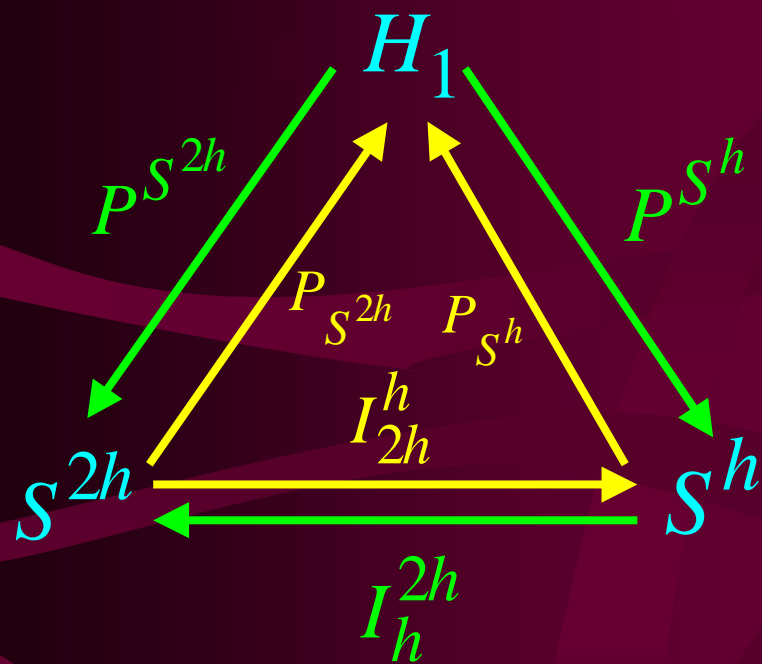
The Good, the Bad, ...
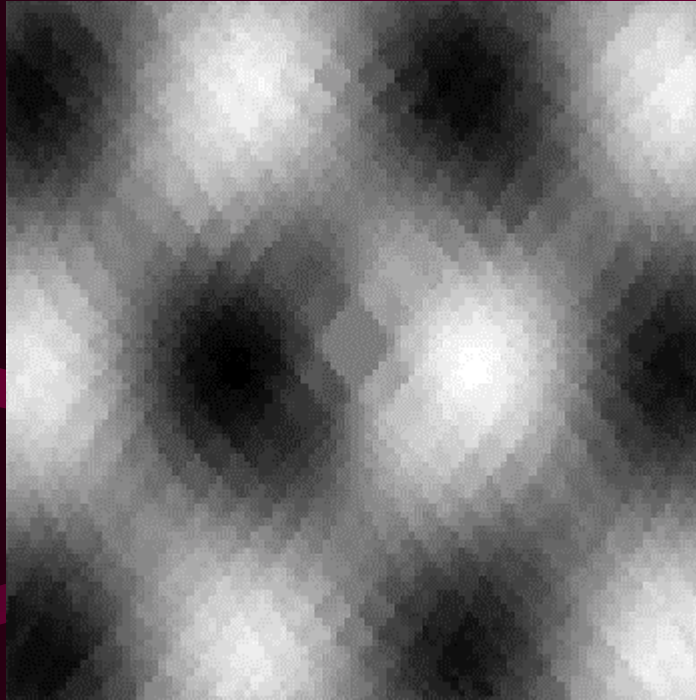
The "good" mode is softly undulating. Has algebraic smoothness and is eliminated from error rapidly.
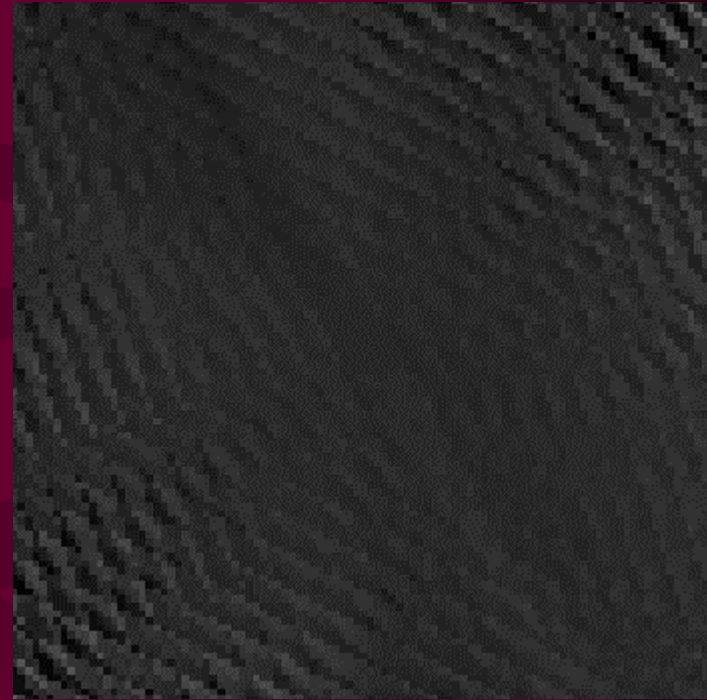
The "bad" mode has some oscillatory behavior, but is predominantly nearly null.

Not eliminated from error.

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Multilevel Projection Method (PML)

To solve $Lu = f$; where $L: H_1 \Rightarrow H_2$.

Intergrid transfer operators defined implicity as whatever operators make the discretization diagram commute, ie.,

I_{2h}^{h} is defined by $P_{S^{2h}} = P_{S^{h}} I_{2h}^{h}$.

I_{h}^{2h} is defined by $P^{S^{2h}} = I_{h}^{2h} P^{S^{h}}$.

Discretization by Projection

The projection discretized problem is:

$$P^{T^h}(L\,P^{S^h}u = f) \quad \text{giving} \quad L^h = P^{T^h}L\,P^{S^h}$$

Define **block subspaces** $S_i^h, \quad i = 1{:}\,p$ so that $S^h = \sum_i S_i^h$

Then: $u^h = \sum_j \alpha_j u_j^h$ (not necessarily a unique representation).

Relaxation $u^h \leftarrow G^h(u^h)$ is defined by:

For $l = 1,2, \dots p,$

 Solve $P^{T_i^h}L^h(u^h + u_i^h) = P^{T_i^h}f^h$ where $u_i^h \in S_i^h$

 Set $u^h \leftarrow (u^h + u_i^h)$

Coarse-grid correction

The coarse-grid correction $u^h \leftarrow CG^h(u^h)$ is defined by:

Solve $P^{T^{2h}} L^h (u^h + P^{S^{2h}} w) = P^{T^{2h}} f^h$

Set $u^h \leftarrow (u^h + P^{S^{2h}} w)$

Hence, a two-grid PML method $u^h \leftarrow PML^h(u^h)$ is given:

1) $u^h \leftarrow G^h(u^h)$

2) $u^h \leftarrow CG^h(u^h)$

In practice, step 2) is replaced with the recursive call

$$u^h \leftarrow PML^{2h}(u^{2h})$$

which gives a PML V-cycle!

PML on Image Reconstruction

Theorem: Let $S^h \equiv \mathbf{span}\{\Psi_j\}_{j=1}^{N}$ so that $S^h = \mathbf{range}(A^*)$.

Then $Bw = f$ is a discretization by orthogonal projection of $Au = f$.

Guts of Proof: For each $j = 1, 2, \ldots N$

$$0 = \left\langle u - P^{S^h} u, \Psi_j \right\rangle = \left\langle u - A^* w, \Psi_j \right\rangle$$

$$= \left\langle u - \sum w_k \Psi_k, \Psi_j \right\rangle = \left\langle u, \Psi_j \right\rangle - \sum w_k \left\langle \Psi_k, \Psi_j \right\rangle$$

$$= Au - Bw.$$

Hence, the orthogonality of the projector requires that

$$Au = Bw$$

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PML on Image Reconstruction

Theorem: Choose $S_i^h \equiv \mathbf{span}\{\Psi_i\}$, the span of the ith strip pixel.

Then PML relaxation is simply point Gauss-Seidel applied to the matrix

equation $Bw = f$.

Theorem: Choose $S^{2h} \equiv \mathbf{span}\{\Psi_j^{2h}\}_{j=1}^{N/2}$, where $\Psi_j^{2h} = \Psi_{2j}^h + \Psi_{2j+1}^h$

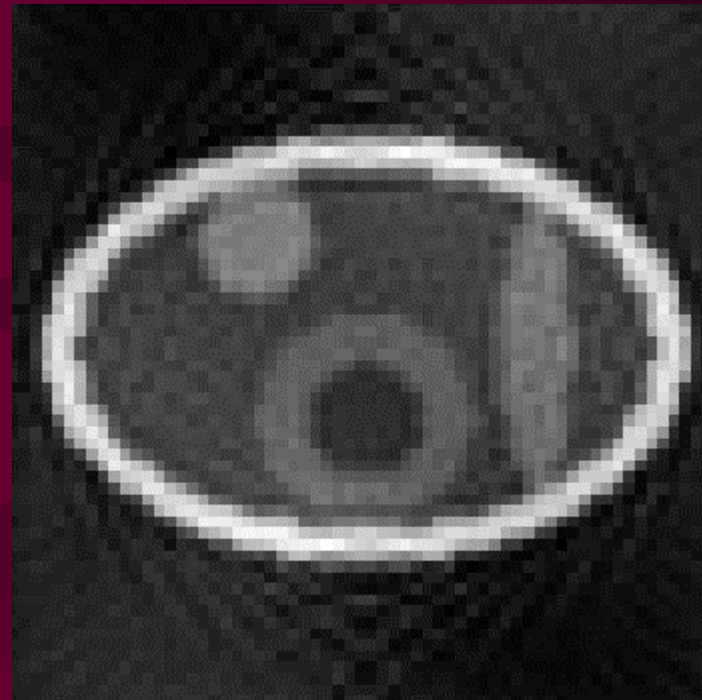(coarse grid: the "fattened" strips by joining adjacent strips).

Then: $I_h^{2h} = \begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \\ & & & & \cdots & \\ & & & & & 1 & 1 \end{pmatrix}_{\frac{N}{2} \times N}$,

$$I_{2h}^h = (I_h^{2h})^T$$

$$B^{2h} = I_h^{2h} B^h I_{2h}^h$$

Hence the standard variational properties hold! Coarse grid operator B has pairs of rows and columns of fine-grid B "lumped" together.
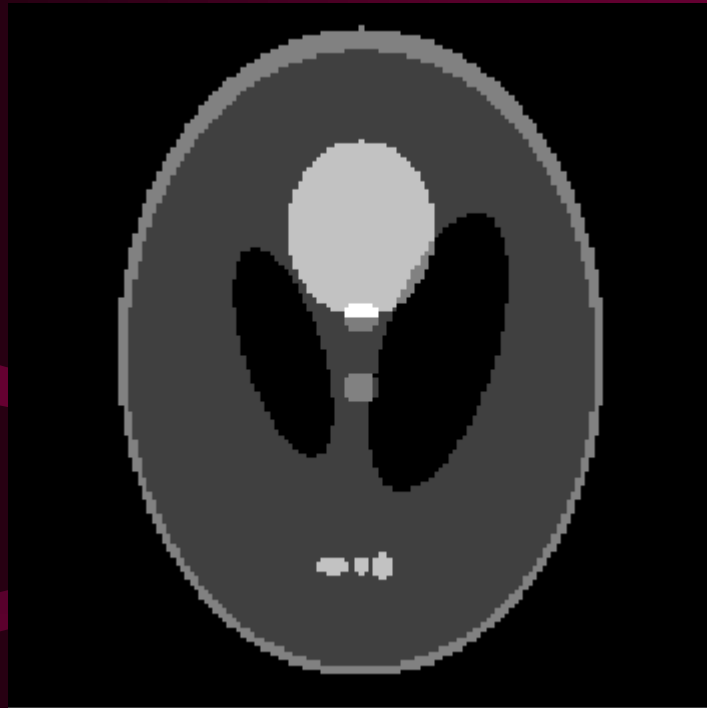
PML Images
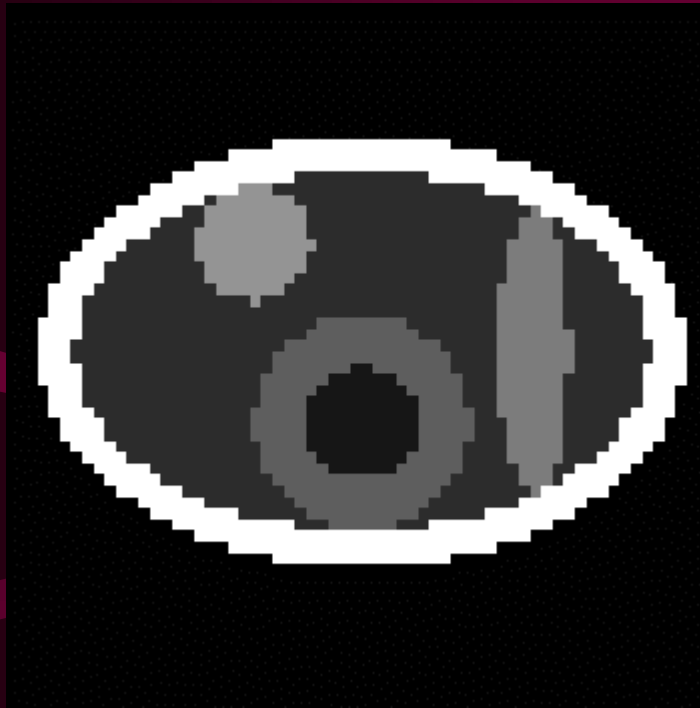
Exact Image

Reconstructed Image

Reconstructed using 3 PML V-cycles, 2 relaxation sweeps downward and 1 relaxation sweep upward. 20 views, 32 strips per view on fine level.
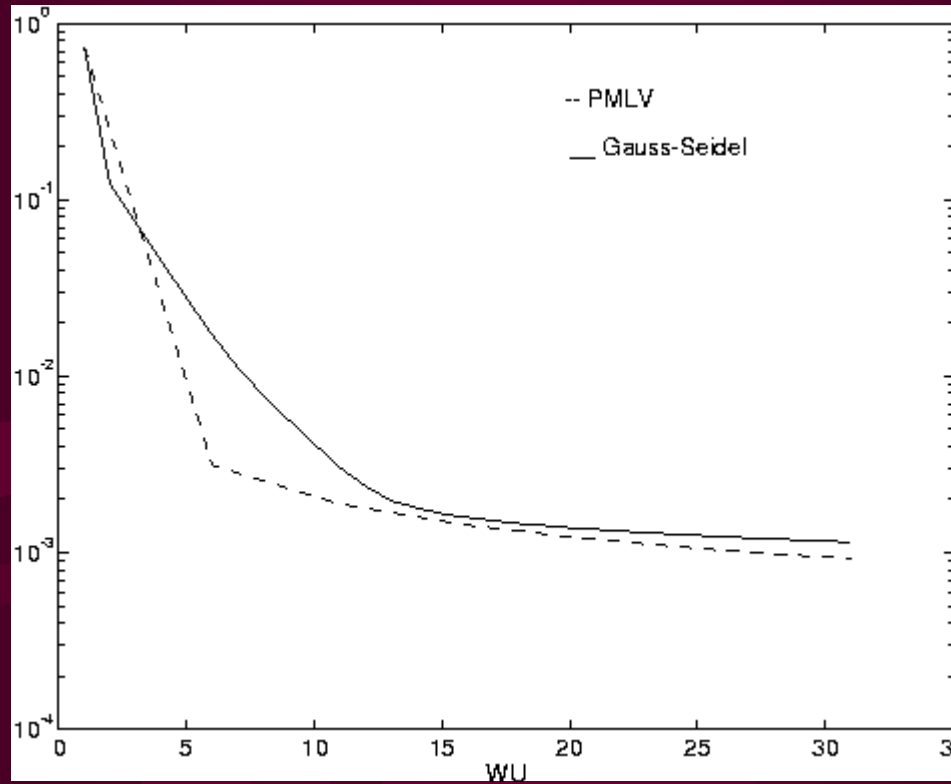
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PML Images

Exact Image

Reconstructed Image

Reconstructed using 3 PML V-cycles, 2 relaxation sweeps downward and 1 relaxation sweep upward. 64 views, 64 strips per view on fine level.

Performance: GS vs. PML

Logarithm of residual norm for Gauss-Seidel on *Bw=f* (solid line) and PML method (dashed line). Plotted against work units (1 WU equals the cost of one relaxation sweep on fine level)

FAC

To do FAC we need a global coarse grid Ω^{2h}, a local refinement grid Ω^h, and a composite grid $\Omega^{\hat{h}}$, which is the combination of the global coarse and local refinement grids.

We also need intergrid transfer operators:

$$I_{\hat{h}}^{h}: \Omega^{\hat{h}} \to \Omega^{h}$$

Composite grid to refinement grid

$$I_{\hat{h}}^{2h}: \Omega^{\hat{h}} \to \Omega^{2h}$$

Composite grid to global coarse grid

$$I_{h}^{\hat{h}}: \Omega^{h} \to \Omega^{\hat{h}}$$

Refinement grid to composite grid

$$I_{2h}^{\hat{h}}: \Omega^{2h} \to \Omega^{\hat{h}}$$

Global coarse grid to composite grid

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FAC

(Fast Adaptive Composite Grid Method)

Once the grids and operators are defined, FAC proceeds in a simple two-step process:

Step 1: $f^{2h} \leftarrow I_{\hat{h}}^{2h}(f^{\hat{h}} - L^{\hat{h}}u^{\hat{h}})$ (restrict the composite residual to global grid)

$u^{2h} = \left[L^{2h}\right]^{-1}f^{2h}$ (solve the global coarse grid error equation)

$u^{\hat{h}} = u^{\hat{h}} + I_{2h}^{\hat{h}}u^{2h}$ (add global correction to composite grid solution)

Step 2: $f^{h} \leftarrow I_{\hat{h}}^{h}(f^{\hat{h}} - L^{\hat{h}}u^{\hat{h}})$ (restrict the composite residual to refinement grid)

$u^{h} = \left[L^{h}\right]^{-1}f^{h}$ (solve the refinement grid error equation)

$u^{\hat{h}} = u^{\hat{h}} + I_{h}^{\hat{h}}u^{h}$ (add refinement correction to composite grid u)
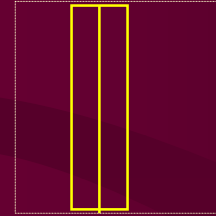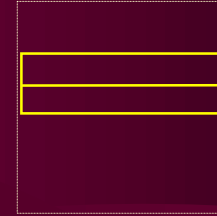
36

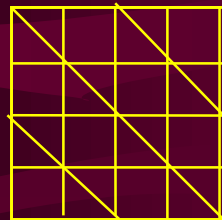# The "Spotlight" Grids
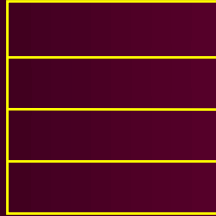
View 1   View 2   View 3   View 1   View 2   View 3

$\Omega^{\hat{h}}$

Ω^{2h}
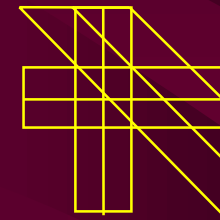
Ω^{h}
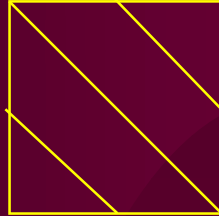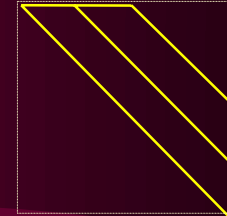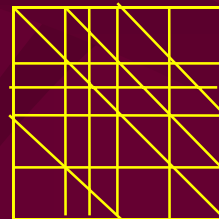
Global
Coarse
Grid

Composite
Grid

Local
Refinement
Grid

Spotlight Tomography

We need to define grid functions $u^{2h}, u^{h}, u^{\widehat{h}}$, as well as operators for the various grids, $B^{2h}, B^{h}, B^{\widehat{h}}$.

Use refinement strips in the same fashion as global coarse strips. Order them after the global grid. Leads to composite grid problem:
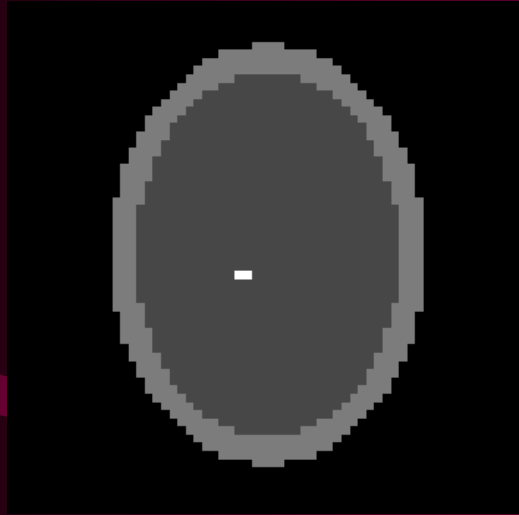
$$B^{\widehat{h}} w^{\widehat{h}} = f^{\widehat{h}}$$ which is $$\begin{pmatrix} B_{2h,2h} & B_{2h,h} \\ B_{h,2h} & B_{h,h} \end{pmatrix} \begin{pmatrix} w^{2h} \\ w^{h} \end{pmatrix} = \begin{pmatrix} f^{2h} \\ f^{h} \end{pmatrix}$$

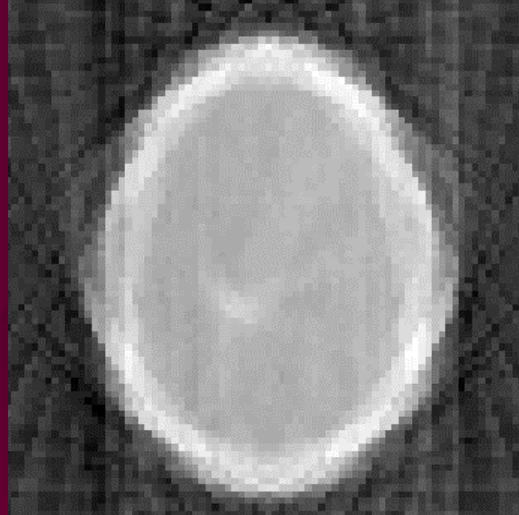FAC implementation: FAC is just **block** Gauss-Seidel on the system!

Step 1: $w^{2h} \leftarrow B_{2h,2h}^{-1}(f^{2h} - B_{2h,h} w^{h})$

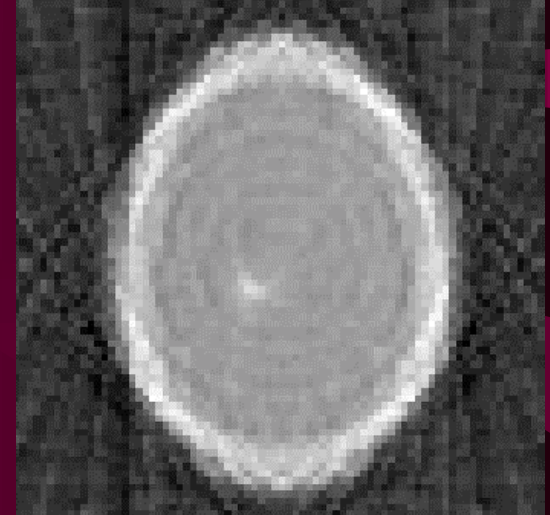**Step 2:** $w^{h} \leftarrow B_{h,h}^{-1}(f^{h} - B_{h,2h} w^{2h})$

Spotlight Tomography
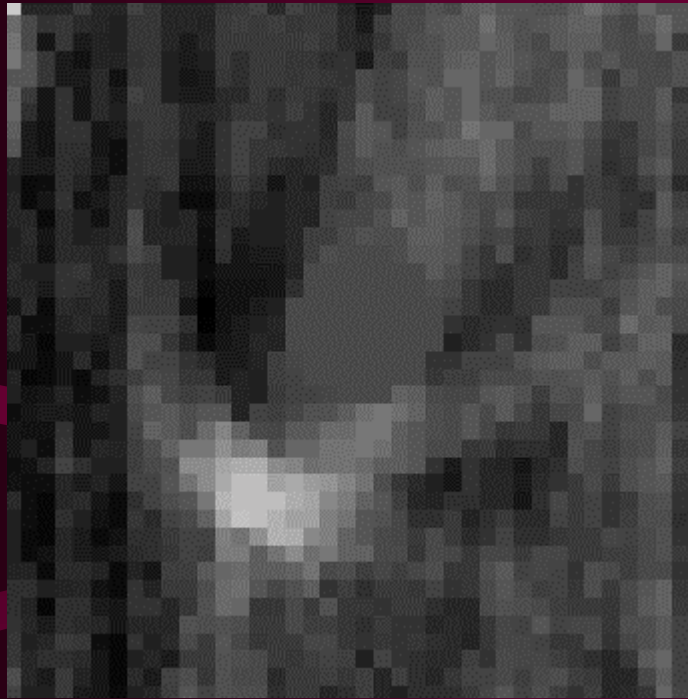

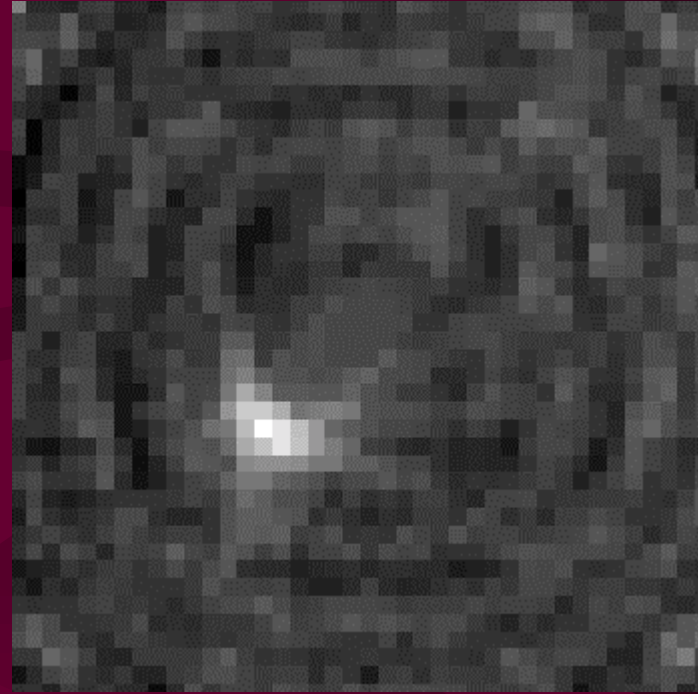Exact solution   PML solution   "Spotlight" solution

PML solution using 20 views, 32 strips per view, *B* is 640x640.

Spotlight solution uses PML strips, plus half-width refinement over the central half of each view. Composite matrix is 1280x1280. Global refinement at same scale requires 2560x2560 matrix.

Spotlight Tomography

Detail of global solution

Detail of spotlight solution

Conclusions

• Natural pixel discretization of the image reconstruction problem leads to iterative methods competitive with ART for quality of image and efficiency.

• Combined with Multilevel Projection Methods, natural pixel discretization yields a multigrid reconstruction algorithm producing quality images faster than other algebraic methods.

• Natural pixels and PML can be combined to perform local refinement of the image, leading to an efficient method of performing "spotlight" tomography.